PERFORMANCE BOUNDS FOR TRANSPORTATION NETWORKS

AN ALGEBRAIC APPROACH

Nadir Farhi
IFSTTAR / GRETTIA

nadir.farhi@ifsttar.fr

Ifsttar - Paris, November 2012.

Outline:

(1) Motivations,

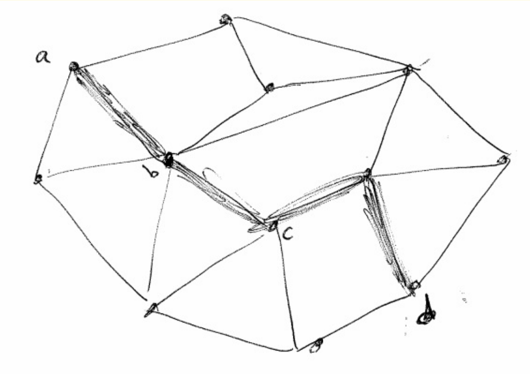
(2) The model (a maximum bound for the travel time through a single-lane road),

(3) Review in network calculus and min-plus algebra,

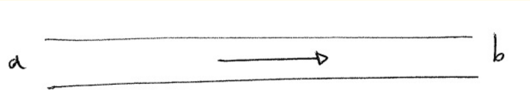
(4) Extension to intersections and networks (a maximum bound for the travel time through a path).

1. Motivation

- Determine a maximum bound for the travel time through a path at a given time, depending on the car-density on the path links.



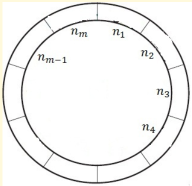
– The first step: determine a maximum bound for the travel time through a single-lane road.

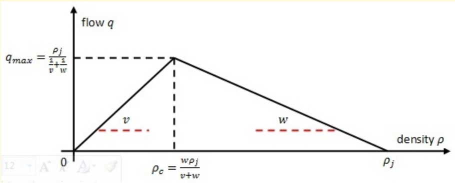


– The approach:

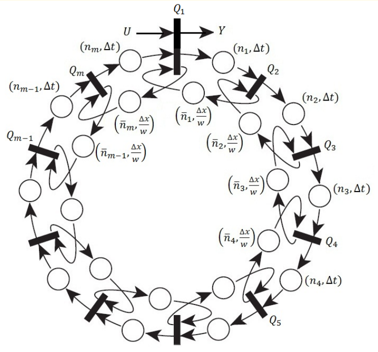
* Adapt an algebraic approach used in communication and computer networks (Chang 2000, Le Boudec Thiran 2001).
* Base on the cell transmission traffic flow model (Daganzo 1994, 1995).

2. The traffic flow model





- $Q_i(t) = \int_0^t q_i(s)ds$: cumulated car outflow from section i up to time t.
- n_i: number of cars in section i at time zero.
- $U(t)$: cumulated inflow up to time t.
- $Y(t)$: cumulated outflow up to time t.



$$Q_1(t) = \min\{U(t), Q_N(t - \Delta x/v) + n_m, Q_2(t - \Delta x/w) + \bar{n}_1\}.$$

$$Q_i(t) = \min\{Q_{i-1}(t - \Delta x/v) + n_{i-1}, Q_{i+1}(t - \Delta x/w) + \bar{n}_i\}, \forall\, i \in \{2,3,\ldots,m-1$$

$$Q_m(t) = \min\{Q_{m-1}(t - \Delta x/v) + n_{m-1}, Q_1(t - \Delta x/w) + \bar{n}_m\}.$$

$$Y(t) = \min\{Q_N(t - \Delta x/v) + n_m, Q_2(t - \Delta x/w) + \bar{n}_1\}.$$

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$$\begin{cases} (f \oplus g)(t) & := min\{f(t), g(t)\} \\ (f * g)(t) & := min\{f(s) + g(t-s)\} \end{cases}$$

$$Q_1 = \gamma^{n_m}\delta^{\Delta x/v} Q_m \oplus \gamma^{\bar{n}_1}\delta^{\Delta x/w} Q_2 \oplus U.$$
$$Q_i = \gamma^{n_{i-1}}\delta^{\Delta x/v} Q_{i-1} \oplus \gamma^{\bar{n}_i}\delta^{\Delta x/w} Q_{i+1}, \quad \forall\, i \in \{2,3,\dots,m-1\}.$$
$$Q_m = \gamma^{n_{m-1}}\delta^{\Delta x/v} Q_{m-1} \oplus \gamma^{\bar{n}_m}\delta^{\Delta x/w} Q_1.$$
$$Y = \gamma^{n_m}\delta^{\Delta x/v} Q_m \oplus \gamma^{\bar{n}_1}\delta^{\Delta x/w} Q_2.$$

$$\gamma^p(t) = \begin{cases} p \text{ if } t = 0 \\ +\infty \text{ for } t > 0 \end{cases} \qquad \delta^T(t) = \begin{cases} 0 \text{ if } t \leq T \\ +\infty \text{ otherwise} \end{cases} \qquad e(t) = \begin{cases} 0 \text{ if } t = 0. \\ +\infty \text{ for } t \geq 0. \end{cases}$$

– Matrix notation A and B are matrices of signals.

$$(A * B)_{ij} := \bigoplus_k A_{ik} * B_{kj}$$

– Then we get the linear dynamics

$$\begin{cases} Q &= A * Q \oplus B * U \\ Y &= C * Q \end{cases}$$

$$A = \begin{pmatrix} \varepsilon & \gamma^{\bar{n}_1}\delta^{\Delta x/w} & \varepsilon & \cdots & \varepsilon & \gamma^{n_m}\delta^{\Delta t} \\ \gamma^{n_1}\delta^{\Delta t} & \varepsilon & \gamma^{\bar{n}_2}\delta^{\Delta x/w} & \varepsilon & \cdots & \varepsilon \\ \varepsilon & \gamma^{n_2}\delta^{\Delta t} & \varepsilon & \gamma^{\bar{n}_3}\delta^{\Delta x/w} & \varepsilon & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \varepsilon \\ \varepsilon & \cdots & \varepsilon & \gamma^{n_{m-2}}\delta^{\Delta t} & \varepsilon & \gamma^{\bar{n}_{m-1}}\delta^{\Delta x/w} \\ \gamma^{\bar{n}_m}\delta^{\Delta x/w} & \varepsilon & \cdots & \varepsilon & \gamma^{n_{m-1}}\delta^{\Delta t} & \varepsilon \end{pmatrix}, \quad B = \begin{pmatrix} e \\ \varepsilon \\ \vdots \\ \vdots \\ \varepsilon \\ \varepsilon \end{pmatrix},$$

$$C = \begin{pmatrix} \varepsilon & \gamma^{\bar{n}_1}\delta^{\Delta x/w} & \varepsilon & \cdots & \varepsilon & \gamma^{n_m}\delta^{\Delta t} \end{pmatrix}$$

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2.1. Initial conditions.

– Initial conditions:

$$Q_i(0) = 0, \forall t \geq 0.$$

– It is sufficient to add (min-plus addition) the signal e:

$$Q_1 = \gamma^{n_m} \delta^{\Delta x/v} Q_m \oplus \gamma^{\bar{n}_1} \delta^{\Delta x/w} Q_2 \oplus U \oplus e.$$

$$Q_i = \gamma^{n_{i-1}} \delta^{\Delta x/v} Q_{i-1} \oplus \gamma^{\bar{n}_i} \delta^{\Delta x/w} Q_{i+1} \oplus e, \ \forall i \in \{2,3,\dots,m-1\}.$$

$$Q_m = \gamma^{n_{m-1}} \delta^{\Delta x/v} Q_{m-1} \oplus \gamma^{\bar{n}_m} \delta^{\Delta x/w} Q_1 \oplus e.$$

$$Y = \gamma^{n_m} \delta^{\Delta x/v} Q_m \oplus \gamma^{\bar{n}_1} \delta^{\Delta x/w} Q_2 \oplus e.$$

$$\begin{cases} Q & = A * Q \oplus B * U \oplus E \\ Y & = C * Q \oplus e \end{cases}$$

– From the min-plus linear system

$$\begin{cases} Q &= A * Q \oplus B * U \oplus E \\ Y &= C * Q \oplus e \end{cases}$$

we get easily

$$Y = CA^*BU \oplus CA^*E \oplus e,$$

– Then

$$Y \geq (CA^*B) * U \oplus (CA^*E \oplus e),$$

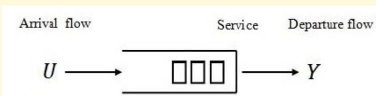
– The curves (CA^*B) and $(CA^*E \oplus e)$ are interpreted as guaranteed service curves in *the network calculus theory*.

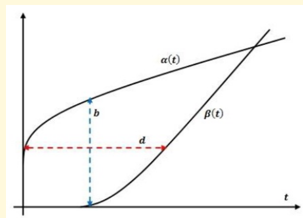
– *service curve:* β is a service curve if

$Y \geq \beta * U$ i.e. $Y(t) \geq min_{0 \leq s \leq t}\{U(s) + \beta(t-s)\}$.

– *arrival curve:* α is an arrival curve if

$U \leq \alpha * U$ i.e. $U(t) - U(s) \leq \alpha(t-s), \forall 0 \leq s \leq t$.





– The virtual delay is bouded:

$$d \leq \sup_{t \geq 0}\{\inf\{h \geq 0, \alpha(t) \leq \beta(t+h)\}\}.$$

- *service couple:* (β, λ) is a service couple if

$$Y \geq \beta * U \oplus \lambda.$$

- We show that the maximum delay is the maximaum horizontal distance between the curves α and $\beta \oplus \lambda$.

- We have from the traffic model:

$$Y \geq (CA^*B) * U \oplus (CA^*E \oplus e).$$

- The for an arrival car-flow $\alpha(t) = rt$ (linear):

$$\tau_{\max}(\rho) = \max\left\{\frac{1}{v}, \quad \frac{\rho}{\rho_j - \rho}\frac{1}{w}, \quad \frac{2\rho}{\rho_j}\frac{1}{w}\right\} m\Delta x.$$

- We have

$$\tau_{avg}(\rho) = \max\left\{\frac{1}{v}, \quad \frac{\rho}{\rho_j - \rho}\frac{1}{w}\right\} m\Delta x.$$

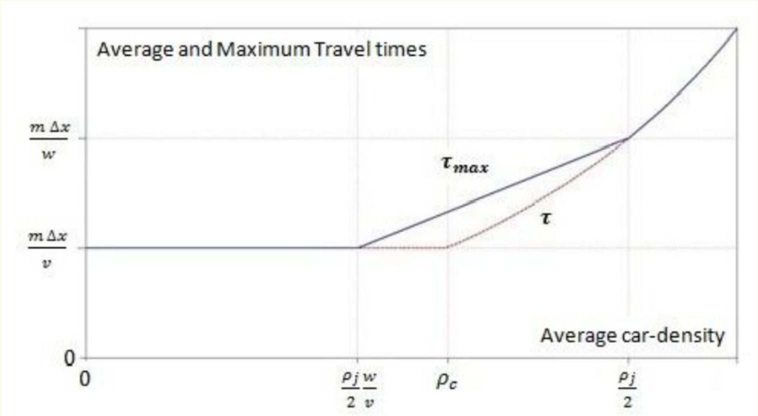
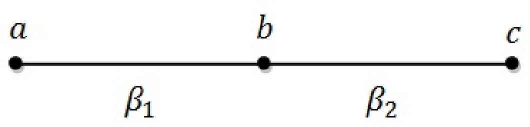


FIGURE 1. The average and the maximum travel times through the road in the case where $w = v/2$.

$$\tau_{\max}(\rho) \geq \tau_{avg}(\rho) \quad \Leftrightarrow \quad \rho \in \left[\frac{\rho_j}{2}\frac{w}{v}\,,\ \frac{\rho_j}{2}\right].$$

3. Road network calculus

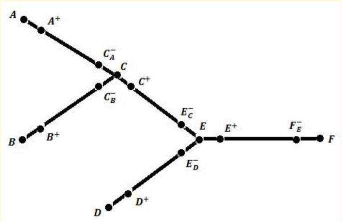
– Series composition fo servers:



– The curve $\beta_1 * \beta_2$ is a service curve for the composed server.

– It remains the calculus of residual services of car-flows passing through controlled intersections.

3.1. Maximum travel time through a path.



(1) Compute the travel times through the roads, independent of the intersections.

(2) Compute the residual services on the intersections. This depends on the control policies. *(not yet done)*

(3) Compose the services in series to get the service on the path.

(4) Deduce a maximum bound for the travel time through the path, for a given arrival.

Main references

(1) Baccelli, F., Cohen, G., Olsder, G. J., Quadrat, J.-P. (1992). Synchronization and Linearity. Wiley.

(2) Chang, C.-S. (2000). Performance guarantees in communication networks. Springer.

(3) Daganzo, C. F. (1994). The cell transmission model: Part I. Trans. Res. B, 28(4) , 269-287.

(4) Farhi, N. Haj-Salem H. & Lebacque J-P. (2013). Algebraic approach for performance bound calculus in transportation networks. TRB 2013.

(5) Farhi, N. (2008). Modlisation minplus et commande du trafic de villes rgulires. PhD thesis, University Paris 1.

(6) Le Boudec, J.-Y., Thiran, P. (2001). Network calculus - A theory for deterministic queuing systems for the internet. Springer-Verlag.